Decentralized Minimum-Energy Coverage Control for Time-Varying Density Functions

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***Abstract*— This paper introduces a minimum-energy approach to the problem of time-varying coverage control. The coverage objective, encoded by a locational cost, is reformulated as a constrained optimization problem that can be solved in a decentralized fashion. This allows the robots to achieve a centroidal Voronoi tessellation by running a decentralized controller even in case of a time-varying density function. We demonstrate that this approach makes no assumptions on the rate of change of the density function and performs the computations in an approximation-free manner. The performance of the algorithm is evaluated in simulation as well as on a team of mobile robots.**

I. INTRODUCTION

Coverage control deals with the problem of distributing a collection of mobile robots in an environment such that the surveillance of its features/events is optimized [1], [2], [3]. The coverage performance of a team of robots over a domain D is typically quantified with respect to a density function, $\phi : q \in D \mapsto \phi(q) \in [0, \infty)$, that encodes the relative importance of the points in such a domain. While many aspects of the coverage problem have been considered in the literature, e.g. limitations on the robots' motion [4], [5], geometric variations on the sensors' footprints [6], [7], or different sensing capabilities [8]; oftentimes the density functions ϕ considered are static and do not depend on time.

However, in some coverage applications, the importance of the points in the domain may evolve over time due to, for example, the tracking of moving targets [9], [10] or new area objectives specified by a human operator [11], [12]. In these cases, it may be advantageous to preserve most of the structure of the coverage problem in [1] and reflect the dynamic nature of these goals by considering the density function to be time-varying, $\phi : (q, t) \in D \times \mathbb{R}_+ \mapsto \phi(q, t) \in \mathbb{R}_+$. Introducing this time dependence, however, has implications on how to design distributed control laws that allow the robots to effectively cover the density function. Past approaches to this problem rely on limitations on the rate of change of the density functions [9] or introduce approximations [11] to produce a distributed controller that optimizes the coverage performance over time.

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In this paper, we propose a constraint-based approach to the time-varying coverage problem that can be executed in an exact, decentralized fashion without imposing any conditions on the rate of change of the density functions. In addition, this constraint-based strategy naturally lends itself composable with additional behaviors that could be concurrently executed by the multi-robot team, e.g. energy saving, collision avoidance [13].

This paper is organized as follows: In Section II, we formally introduce the problem of multi-robot coverage and discuss some of the existing strategies for time-varying coverage control. In Section III, we introduce the technical details of the constraint-based task execution framework. Using these results, the proposed strategy for time-varying coverage control is presented in Section IV. This algorithm is implemented on a real robotic platform and its performance is compared to other approaches in Section V. Section VI concludes the paper.

II. COVERAGE CONTROL WITH TIME-VARYING DENSITY FUNCTIONS

A. Coverage control

Consider a team of N robots, whose positions are denoted by $x_i \in \mathbb{R}^d,\ i \in \{1, \dots, N\}$, where $d = 2$ for planar robots and $d = 3$ in the case of aerial robots. The objective of the coverage control problem is to distribute this team of mobile robots in a domain $D \subset \mathbb{R}^d$ with respect to a density function that encodes the relative importance of the points in D, $\phi : D \to [0, \infty)$, where $\phi(q) = 0$ only for a finite collection of points. As shown in [14], one natural choice is to make Robot i, $i \in \{1, \dots, N\} := \mathcal{N}$, be in charge of covering the points that are closest to it,

$$V_i(x) = \{p \in D \mid \|p - x_i\| \le \|p - x_j\|,\ \forall j \in \mathcal{N}\},$$

that is, its Voronoi cell. The quality of coverage of the multi-robot team can then be encoded through the cost [1],

$$\mathcal{H}(x) = \sum_{i=1}^{N} \int_{V_i(x)} \|x_i - q\|^2 \phi(q)\, \mathrm{d}q, \qquad (1)$$

with a lower value of the cost corresponding to a better coverage. Let

$$G_i(x) = \frac{\int_{V_i(x)} q\phi(q)\, \mathrm{d}q}{\int_{V_i(x)} \phi(q)\, \mathrm{d}q} \qquad (2)$$

be the center of mass of the Voronoi cell of Robot i. A necessary condition for (1) to be minimized is that the position of each robot corresponds to the center of mass of its Voronoi cell [15], that is, the robots are in a *centroidal Voronoi tessellation* (CVT).

In order to approach the centroidal Voronoi tessellation, we can make the robots follow a direction of descent of the type

$$x^{(k+1)} = x^{(k)} - \alpha_k \frac{\partial J}{\partial x}(x^{(k)}),$$

where the superscript k denotes the time-step and $J(x)$ is a function whose stationary points are the centroidal Voronoi tessellations [15]. A natural choice for J is

$$J(x) = \sum_{i=1}^{N} \frac{1}{2}\|x_i - G_i(x)\|^2 = \sum_{i=1}^{n} J_i(x). \tag{3}$$

Taking the derivative of J_i with respect to x_i, one obtains,

$$\frac{\partial J_i}{\partial x_i} = (x_i - G_i(x))^T \left(I - \frac{\partial G_i(x)}{\partial x_i}\right), \tag{4}$$

where I is the identity matrix. Note that, even if $G_i(x)$ depends on the entire ensemble state of the robotic swarm, x, Robot i only requires information about its Delaunay neighbors to compute it. Thus, the gradient in (4) can be calculated in a decentralized fashion.

B. Time-Varying Densities

The formulation of the coverage control problem in (1) considers a static density function, $\phi(q)$, over the domain of interest, that is, the relative importance of the points does not change over time. In situations where the importance of the points in the domain may vary with time, however, the density function of the domain is time-variant. Considering a time-varying density function $\phi : (q,t) \in D \times \mathbb{R}_+ \mapsto \phi(q,t) \in \mathbb{R}_+$, results in the following locational cost,

$$\mathcal{H}(x,t) = \sum_{i=1}^{N} \int_{V_i(x)} \|q - x_i\|^2 \phi(q,t)\, \mathrm{d}q. \tag{5}$$

Assuming that the variation on the density function over time was quasi-static [9], a control law that minimizes (5) is

$$u_i = \dot{G}_i(x,t) - \left(\kappa + \frac{\dot{M}_i(x,t)}{M_i(x,t)}\right)(x_i - G_i(x,t)), \tag{6}$$

with $M_i(x,t) = \int_{V_i(x)} \phi(q,t)\, \mathrm{d}q$, $G_i(x,t)$ defined as in (2) with the time-varying density, $\phi(q,t)$, instead. The time derivatives in (6) are computed as follows,

$$\dot{M}_i = \int_{V_i} \dot{\phi}(q,t)\, \mathrm{d}q, \quad \dot{G}_i = \frac{1}{M_i}\left(\int_{V_i} q\dot{\phi}(q,t)\, \mathrm{d}q - M_i G_i\right),$$

where we have suppressed the dependencies M_i, G_i and their time derivatives on (x,t) and the dependency of V_i on x for notational convenience.

The restrictiveness of the quasi-static approach in [9] motivated a different approach in [11]. As illustrated in [11], [16], considering the time-varying version of the cost in (3),

$$J(x,t) = \sum_{i=1}^{N} \frac{1}{2}\|x_i - G_i(x,t)\|^2 = \sum_{i=1}^{n} J_i(x,t), \tag{7}$$

one can achieve a CVT, without imposing conditions on the variation of $\phi(q,t)$, by setting

$$u = \left(I - \frac{\partial G}{\partial x}\right)^{-1} \left(\kappa(G(x,t) - x) + \frac{\partial G}{\partial t}\right), \tag{8}$$

where $G = [G_1^T, \dots, G_N^T]^T$.

However, inverting the matrix $I - \frac{\partial G}{\partial x}$ in (8) cannot be done in a decentralized fashion. For this reason, in [11], the inverse is approximated by a truncated Neumann series as

$$\left(I - \frac{\partial G}{\partial x}\right)^{-1} \approx I + \frac{\partial G}{\partial x} \tag{9}$$

which allows each robot to evaluate its corresponding term based solely on information about its Delaunay neighbors. As discussed in Section I, this paper presents a decentralized solution to the time-varying coverage control problem which does not require us to make any such approximations. Next, we introduce some of the tools necessary to develop such an algorithm.

III. TECHNICAL BACKGROUND

This paper uses the constraint-based task execution framework introduced in [13] to perform coverage control in the presence of time-varying density functions. Consequently, this section introduces some of the tools required to develop the proposed algorithm which will be presented in Section IV.

The execution of a task by a robot can be encoded using the following pointwise minimum-energy constrained optimization problem,

$$\min_u \|u\|^2 \quad \text{s.t.} \quad c_{task}(x,u) \geq 0,$$

where u is the control effort expended by the robot, x is its state, and c_{task} symbolizes a constraint function which ensures the execution of the task. Such a constraint-based formulation is advantageous in terms of its suitability for long-term autonomy applications as well as composability with other tasks that need to be performed [17], [18]. The initial formulation in this section considers constraints that do not explicitly depend on time. Later in the section, the time-varying formulation is presented.

The feasibility of this task execution framework is ensured by the introduction of slack variables in the constraint:

$$\begin{aligned} &\min_{u,\delta} \|u\|^2 + |\delta|^2 \\ &\text{s.t.} \quad c_{task}(x,u) \geq -\delta, \end{aligned} \tag{10}$$

where δ is the slack variable and signifies the extent to which the task constraint can be violated. An effective way of enforcing such constraints in a multi-robot system is to use control barrier functions, which are introduced next.

A. Control Barrier Functions

Consider a dynamical system in control affine form,

$$\dot{x} = f(x) + g(x)u,$$

where $x \in \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, with f and g being Lipschitz continuous vector fields. Consider a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, and define the *safe set* S as its zero-superlevel set:

$$S = \{x \in \mathbb{R}^n \mid h(x) \geq 0\}. \tag{11}$$

The function h is called a *(zeroing) control barrier function (CBF)* if the following condition is satisfied:

$$\sup_{u \in U} \{L_f h(x) + L_g h(x) u + \alpha(h(x))\} \geq 0 \quad \forall x \in \mathbb{R}^n, \tag{12}$$

where α is a locally Lipschitz extended class $\mathcal{K}$ function [19], and $L_f h(x)$ and $L_g h(x)$ denote the Lie derivatives of h in the directions f and g, respectively. The following theorem from [19], [13] summarizes two important properties of zeroing CBFs.

Theorem 1. *Given a dynamical system in control affine form $\dot{x} = f(x) + g(x)u$, where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ denote the state and the input, respectively, f and g are locally Lipschitz, and a set $S \subset \mathbb{R}^n$ defined by a continuously differentiable function h as in* (11)*, any Lipschitz continuous controller u such that* (12) *holds renders the set S forward invariant and asymptotically stable, i. e.,:*

$$x(0) \in S \Rightarrow x(t) \in S \; \forall t \geq 0$$
$$x(0) \notin S \Rightarrow x(t) \to\in S \text{ as } t \to \infty,$$

where $x(0)$ denotes the state x at time $t = 0$ and the notation $x(t) \to\in S$ indicates that $x(t)$ asymptotically approaches the set S.

Proof. See [19] and [13]. □

In this paper, we encode the execution of the time-varying coverage control task via a zeroing CBF-based constraint for each robot. Consequently, the zeroing CBFs themselves explicitly depend on time. To this end, the definition of zeroing CBFs given in [19] is extended for the time-varying case.

Definition 2 (Time-Varying CBFs [20]). *Given a function $h : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$, continuously differentiable in both its arguments, consider a dynamical system in control affine form $\dot{x} = f(x) + g(x)u$, where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ denote system state and input, respectively, f and g are locally Lipschitz, and the set $S = \{x \in \mathbb{R}^n \mid h(x,t) \geq 0\}$. The function h is a time-varying zeroing CBF defined on $\mathbb{R}^n \times \mathbb{R}_+$, if there exists a locally Lipschitz extended class $\mathcal{K}$ function α such that, $\forall x \in \mathbb{R}^n$, $\forall t \in \mathbb{R}_+$,*

$$\sup_{u \in U} \left\{ \frac{\partial h}{\partial t} + L_f h(x,t) + L_g h(x,t)\, u + \alpha(h(x,t)) \right\} \geq 0. \tag{13}$$

We now demonstrate how CBFs can be incorporated into the constrained optimization problem (10) to accomplish the execution of robot tasks.

B. Minimum-Energy Gradient Descent

The execution of tasks which involve the minimization of a cost function J—such as the coverage control task investigated in this paper—can be achieved by generating a control signal $u(t)$ using the optimization problem

$$\min_u J(x), \tag{14}$$

where x and u are coupled through the single integrator dynamics $\dot{x} = u$. In [13], we show that solving (14) in order to synthesize $u(t)$ is equivalent to solving the following constraint-based optimization problem, in the sense that they both achieve the goal of minimizing the cost J:

$$\begin{aligned} \min_{u,\delta} \quad & \|u\|^2 + |\delta|^2 \\ \text{s.t.} \quad & \frac{\partial h}{\partial x} u \geq -\alpha(h(x)) - \delta \end{aligned} \tag{15}$$

where $\delta \in \mathbb{R}$ is the slack variable signifying the extent to which the task constraint can be violated, α is an extended class $\mathcal{K}$ function, and $h(x) = -J(x)$ is a zeroing CBF. The zero-superlevel set of h is $S = \{x \mid h(x) \geq 0\} = \{x \mid J(x) \leq 0\} = \{x \mid J(x) = 0\}$, where the last equality holds because the cost $J(x)$ is a non-negative function.

The following proposition, proved in [13], establishes how the constraint-based optimization problem given in (15), allows the accomplishment of the task encoded by $J(x)$.

Proposition 3. *The solution of the optimization problem* (15)*, where $h(x) = -J(x)$ and α is an extended class $\mathcal{K}$ function, solves* (14)*, driving the state x of the dynamical system $\dot{x} = u$ to a stationary point of the cost J.*

In fact, for the special case when J is strictly convex and $J(0) = 0$, we have that

$$\frac{\partial J}{\partial x}(x) \neq 0, \; \forall x \neq 0.$$

Consequently, using Theorem 1 we get $x \to\in S$, i. e. $J(x(t)) \to 0$, as $t \to \infty$.

Using the above described formulation, this paper encodes the problem of covering a time-varying density function as a constraint-based optimization problem in Section IV. But first, we discuss the conditions under which the optimization problem in (15) can be solved in a decentralized fashion.

C. Decentralized Constraint-Based Control of Multi-Robot Systems

Assume that each robot in the multi-robot team is able to measure the relative positions of a subset of the robot team as described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, N\}$ is the set of vertices of the graph, representing the robots, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges between the robots, encoding adjacency relationships. For example, the adjacency relationships for the multi-robot coverage control task investigated in this paper is described by a Delaunay graph [1].

Let $x = [x_1^T, \ldots, x_N^T]^T \in \mathbb{R}^{Nd}$ denote the ensemble state of the multi-robot team. As the robots are solving a time-varying coverage control problem, we consider a time-varying total cost $J(x,t)$. Then, a general expression for this cost that leads to decentralized control laws [16] is given by

$$J(x,t)=\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}J_{ij}(\|x_i-x_j\|,t), \tag{16}$$

where $\mathcal{N}_i$ is the neighborhood set of Robot i, and $J_{ij}:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, $J_{ij}(\|x_i-x_j\|,t)=J_{ji}(\|x_j-x_i\|,t)$ is a symmetric, pairwise cost between robots i and j. We assume that $J_{ij}(x,t)\geq 0,\ \forall(i,j)\in\mathcal{E},\ \forall x\in\mathbb{R}^n, t\in R_+$, so that $J(x,t)\geq 0,\ \forall x\in\mathbb{R}^n, t\in\mathbb{R}_+$. It should be noted that (7) can be written in the form of (16) as a consequence to the graph topology induced by the Voronoi partition.

The following proposition outlines the optimization problems whose solutions lead to a decentralized minimization of the cost $J(x,t)$ in (16).

Proposition 4 (Constraint-driven decentralized time-varying task execution)**.** *Given the time-varying pairwise cost function J defined in* (16)*, a collection of N robots, obeying single integrator dynamics, minimizes J in a decentralized fashion, if each robot executes the control input, solution of the following optimization problem:*

$$\begin{aligned}&\min_{u_i,\delta_i}\ \|u_i\|^2+|\delta_i|^2\\ &\text{s.t.}\ -\frac{\partial J_i}{\partial x_i}u_i\geq-\alpha(-J_i(x))+\frac{\partial J_i}{\partial t}-\delta_i,\end{aligned} \tag{17}$$

where $J_i(x,t)=\sum_{j\in\mathcal{N}_i}J_{ij}(\|x_i-x_j\|,t)$ and α is an extended class $\mathcal{K}$ function, $\alpha:x\in\mathbb{R}\mapsto\alpha(x)\in\mathbb{R}$, superadditive for $x<0$, i.e. $\alpha(x_1+x_2)\geq\alpha(x_1)+\alpha(x_2),\ \forall x_1,x_2<0$.

Proof. Using (13) from Definition 2, the proof follows similar to [13]. ☐

We are now ready to present a novel approach for executing decentralized approximation-free coverage control under time-varying density functions using a team of robots.

IV. AN EXACT AND DECENTRALIZED APPROACH TO TIME-VARYING COVERAGE

As described in Section II, effective coverage of a domain can be achieved by driving the robots to the stationary points of the time-varying cost functional $J(x,t)$ given in (7), which correspond to the CVT. To this end, we allow each Robot i to solve the optimization problem presented in (17). Plugging in expressions for the partial derivatives of $J_i(x,t)$ as pertaining to the time-varying coverage control problem, (17) yields,

$$\begin{aligned}&\min_{u_i,\delta_i}\ \|u_i\|^2+|\delta_i|^2\\ &\text{s.t.}\ -(x_i-G_i(x,t))^T\left(I-\frac{\partial G_i(x,t)}{\partial x_i}\right)u_i\\ &\qquad\geq-\alpha(-J_i(x,t))-(x_i-G_i(x,t))^T\frac{\partial G_i(x,t)}{\partial t}-\delta_i,\end{aligned} \tag{18}$$

which is both exact and decentralized. The partial derivatives of the $G_i(x,t)$ are given by [21],

$$\frac{\partial G_i(x,t)}{\partial x_i}=\sum_{j\in\mathcal{N}_i}\frac{\int_{\partial V_{ij}(x)}(q-G_i(x,t))\phi(q,t)(q-x_i)^T\,\mathrm{d}q}{m_i(x,t)\|x_j-x_i\|},$$

$$\frac{\partial G_i(x,t)}{\partial t}=\frac{\int_{V_i(x)}(q-G_i(x))\frac{\partial\phi(q,t)}{\partial t}\,\mathrm{d}q}{m_i(x,t)},$$

with $m_i(x,t)=\int_{V_i}\phi(q,t)\,\mathrm{d}q$ the mass in the Voronoi cell of Robot i.

Proposition 5. *Consider a team of N single-integrator robots, tasked with covering a region as specified by a time-varying importance density function. Under u^*, solution of* (18)*, where α is a superlinear extended class $\mathcal{K}$ function, the robots achieve a CVT.*

Proof. From Proposition 4, we know that executing u^* will drive the robots towards a stationary point of the cost function J. As discussed in Section 5.4 of [15], any search algorithm that attains the stationary points of the cost J, achieves a CVT configuration. ☐

We further demonstrate that, under the assumption that the robots do not have actuator limitations, the optimization problem presented in (18) can be reformulated to exclude slack variables in the optimization problem.

Proposition 6. *Consider a team of N single-integrator robots with no actuator constraints, i.e., $U=\mathbb{R}^m$, tasked with covering a region as specified by a time-varying importance density function. Let each robot solve the following problem:*

$$\begin{aligned}&\min_{u_i}\ \|u_i\|^2\\ &\text{s.t.}\ -(x_i-G_i(x,t))^T\left(I-\frac{\partial G_i(x,t)}{\partial x_i}\right)u_i\\ &\qquad\geq-\alpha(-J_i(x,t))-(x_i-G_i(x,t))^T\frac{\partial G_i(x,t)}{\partial t}.\end{aligned} \tag{19}$$

Under the control action generated by this optimization problem, the robots achieve a centroidal Voronoi tessellation (CVT).

Proof. The total time derivative of the time-varying coverage cost function $J(x,t)$ given in (7) can be expressed as,

$$\begin{aligned}\dot{J}&=\sum_{i=1}^{N}\dot{J}_i(x)=\sum_{i=1}^{N}\left(\frac{\partial J_i}{\partial x_i}u_i+\frac{\partial J_i}{\partial t}\right)\\ &=\sum_{i=1}^{N}(x_i-G_i(x,t))^T\left(I-\frac{\partial G_i(x,t)}{\partial x_i}\right)u_i\\ &\quad-\sum_{i=1}^{N}(x_i-G_i(x,t))^T\frac{\partial G_i(x,t)}{\partial t}.\end{aligned}$$

Consequently, given the superadditivity property of α and summing over the constraints corresponding to each Robot

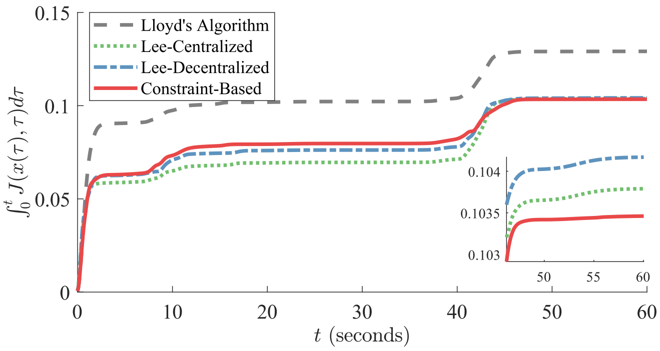

Fig. 1. Evolution of the integral of the cost, $J(x, t)$, over time for the proposed constraint-based approach, Lloyd's algorithm [14] as well as the centralized and decentralized controllers in [11]. The final value of the cumulative cost for the proposed algorithm is very similar (although slightly lower) to the controllers that consider the effects of the time-varying densities in [11]. Ignoring the effects of a time-varying density function causes an appreciable difference in the case of Lloyd's algorithm. Inset highlights differences in costs towards the end of the experiment.

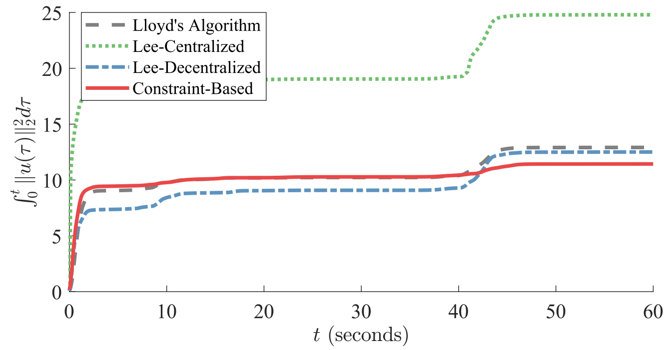

Fig. 2. Comparison of the cumulative control effort for the proposed constraint-based approach, Lloyd's algorithm [14] as well as the centralized and decentralized controllers in [11]. While the performance of the algorithms in [11] are similar to the proposed approach in terms of the final cost (see Fig. 1), the control effort required for the team to track the density functions is higher.

$i \in \mathcal{N}$ in the optimization problem (19), we get,

$$\dot{J}(x, t) \leq \alpha(-J(x, t))$$

Let $\bar{\alpha}(r) = -\alpha(-r)$. Then, by the properties of extended class $\mathcal{K}$ functions, $\bar{\alpha}$ is also an extended class $\mathcal{K}$ function. Thus,

$$\dot{J}(x, t) \leq -\bar{\alpha}(J(x, t)),$$

and thus, by applying the comparison lemma [22], one can observe that:

$$J(t) \leq \beta(J(x_0, 0), t),$$

with β a class $\mathcal{KL}$ function and x_0 the configuration of the robots at time $t = 0$. Therefore, $J(x, t) \to 0$ as $t \to \infty$, that is, the system converges to a CVT since $J(x, t) = 0 \Leftrightarrow x_i(t) = G_i(x, t), \forall i \in \mathcal{N}$. □

V. SIMULATIONS AND EXPERIMENTAL RESULTS

The performance of the proposed constraint-based approach is evaluated in simulation as well as on a team of differential drive robots on the Robotarium [23], a remotely accessible multi-robot testbed at the Georgia Institute of Technology. The experiment, uploaded via web, is remotely executed on the Robotarium and the data is made available to the user once the experiment is finalized. On each control iteration, the Robotarium provides the poses of the robots involved in the experiment and allows the user to specify the linear and angular velocities of each robot in the team.

The proposed constraint-based controller in (18) is compared in simulation with the standard Lloyd's algorithm [14], whereby $\dot{x}_i = \kappa(G_i(x) - x_i), \kappa > 0, \forall i \in \mathcal{N}$; and with the centralized strategy in (8) from Lee *et al.* [11] as well as with the decentralized variant that uses the Neumann approximation in (9). In order to minimize the influence of the proportional gain, the simulation parameter $\kappa = 1$ was chosen for all three controllers. In the case of the proposed controller, the extended class $\mathcal{K}$ function was $\alpha(x) = x^{\frac{1}{3}}$.

The simulations are implemented on the Robotarium simulator with the objective of providing a realistic framework that considers robot dynamics and actuator bounds, thus providing a fair comparison between the different algorithms.

As presented in Sections II and IV, the considered coverage control algorithms assume that the robots move according to single integrator dynamics. However, the robots considered in this section have a differential drive kinematic configuration, whose movement is best described by the so-called unicycle dynamics,

$$\dot{x}_i = [v_i \cos\theta_i, v_i \sin\theta_i]^T, \quad \dot{\theta}_i = \omega_i,$$

where θ_i is the orientation of the robot. The control inputs v_i and ω_i are the linear and angular velocities, which can be calculated using the near-identity diffeomorphism in [24].

We consider the following time-varying density function to be covered by a team of 6 differential drive robots over a time interval of 60 seconds,

$$\begin{aligned}\phi_{exp}(q,t) &= 1 \\ &+ 10^3 \frac{1-\sin\left(2\pi 10^{-3} t\right)}{2} e^{\left(-\frac{(q_x+0.2)^2-(q_y+0.1)^2}{0.4}\right)} \\ &+ 10^3 \frac{1-\sin\left(2\pi 10^{-3} t - \frac{\pi}{2}\right)}{4} e^{\left(-\frac{(q_x-0.6)^2-(q_y-0.2)^2}{0.1}\right)}.\end{aligned} \tag{20}$$

In order to compare the performance of the different algorithms, one can compute the integral of the cost $J(x, t)$ over time [11], as a metric of how well the density function is being tracked by the robot team,

$$\int_0^t J(x(\tau), \tau)\, \mathrm{d}\tau.$$

As it can be observed in Fig. 1, considering the effects of the time-varying density makes our approach and the controllers in [11] outperform Lloyd's algorithm, which was designed for the time-invariant case. However, while producing similar coverage of the density function, some algorithms may require higher control efforts from the robots than others. Therefore, we use the following metric to measure the

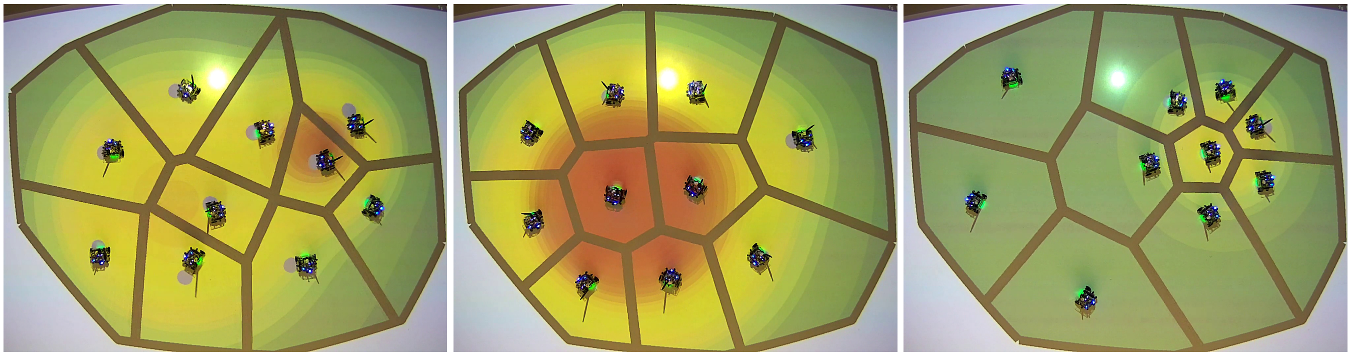

Fig. 3. Snapshots from the time-varying coverage control experiment deployed on a multi-robot team operating on the Robotarium [23]. The time-varying density function is depicted by projecting its contour plot onto the testbed. As seen, using the constraint-based coverage algorithm, the robots track the centroids of their Voronoi cells, depicted as gray circles.

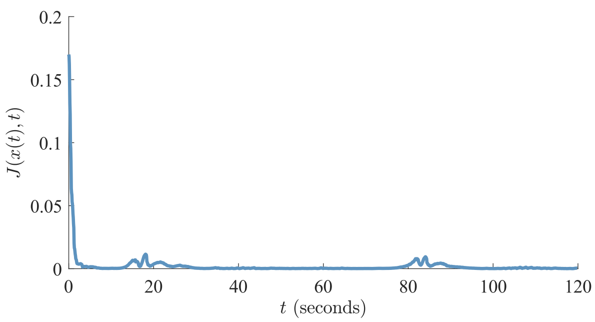

Fig. 4. Evolution of the cost $J(x, t)$ for the proposed minimum energy coverage algorithm. The constraint-based approach drives the robots in a direction which reduces the overall coverage cost considered in (7) to zero. The temporary increases in the cost can be attributed to the fact that the robots have actuator constraints and thus cannot track arbitrarily high velocities generated by the optimization program. As the robots reduce the distance from the moving centroids of their Voronoi cells, the cost goes back towards zero.

amount of energy used by the robot team to cover the density function,

$$\int_0^t \|u(\tau)\|_2^2 \, \mathrm{d}\tau.$$

Figure 2 shows the control effort expended by the robots when executing the different algorithms considered. While the approaches from [11] produced similar cumulative costs in Fig. 1, we can observe that the control effort demanded by these controllers is higher than that of the proposed strategy in this paper.

Figure 3 shows a series of snapshots of an experiment executed on the Robotarium. Ten GRITSBots were deployed to cover the density function in (20) for a total duration of 2 minutes. We can observe how the robots effectively track the centroids of their Voronoi cells as the density function changes over time. The evolution of the cost, $J(x, t)$, for this experiment is shown in Fig. 4, where the cost is kept close to zero. The temporary increases in the cost around $t = 19$s and $t = 85$s are due to the actuator constraints of the robots, which limit their ability to maintain a CVT during rapid changes of the density function.

VI. CONCLUSIONS

This paper develops an exact and decentralized algorithm for the multi-robot time-varying coverage control problem. In our approach, the coverage objective is encoded as a constraint in a minimum-energy optimization program executed by each robot. Slack variables encoded within the constraint ensure feasibility of the optimization program. The performance of our algorithm is compared with other approaches to demonstrate how the constraint-based method effectively covers a region with time-varying importance densities in a decentralized and approximation-free manner.

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