Sensor Coverage Control Using Robots Constrained to a Curve

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***Abstract*— In this paper we consider a constrained coverage control problem for a team of mobile robots. The robots are asked to provide sensor coverage over a two-dimensional domain, while being constrained to only move on a curve. The unconstrained coverage problem can be effectively solved by defining a locational cost to be minimized by the robots, in a decentralized fashion, using gradient descent. However, a direct projection of the solution to the unconstrained problem onto the curve may result in a very poor spatial allocation of the team within the two-dimensional domain. Therefore, we propose a modification to the locational cost, which incorporates the constraints, and a convex relaxation that allows us to efficiently minimize a convex approximation of the cost using a decentralized strategy. The resulting algorithm is implemented on a team of mobile robots.**

I. INTRODUCTION

The proliferation of robots operating in highly structured environments can be attributed to the limited uncertainty that such surroundings provide, which helps simplifying the motion planning and control of the robots working in them, e. g. [1], [2], [3], [4]. Although building the infrastructure may be a high price to pay in order to reap such benefits, applications for which the infrastructure is already in place constitute the perfect scenario for robots to exploit. A prominent case is that of wire-traversing robots, a technology used in applications such as power transmission line maintenance [5], environmental monitoring [6] or agriculture robotics [7].

For wire-traversing robots, the wires or cables are not only essential for locomotion, but also act as passive supports which are particularly good, from an energetic perspective, for surveillance tasks. In fact, as opposed to aerial robotic platforms, such as quadcopters, which demand high amounts of energy to operate, the low-energy consumption of wire-traversing robots makes them ideal candidates for long-term tasks such as environmental monitoring [8]. And even fixed-wing aerial robots, which are less energy consuming that rotary-wing robots, are preferable over wire-traversing robots only when the environment to monitor is sufficiently big and stationary [8]. However, while significant amount of attention has been paid to the mechanical design of wire-traversing robots [9], [10], by developing mechanisms that enable maneuvers such as obstacle avoidance [11] or wire transfer [12], scant progress has been made on designing control strategies that are suitable for environmental monitoring using robots constrained to move on wires [13].

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Coverage control constitutes an effective strategy for environmental monitoring applications, as it deals with how to distribute a collection of agents in the environment with respect to an underlying density function, which represents areas of interest within the domain [14]. A significant body of work has been developed in the context of coverage control, formulated initially for a convex domain with homogeneous agents, e. g. [14]. This context has been subsequently modified to capture other assumptions, such as non-convex [15], [16] or non-euclidean [17] environments, or introducing heterogeneous teams of robots [18], [19], [20], among others. However, the problem of covering a planar environment while being constrained to move in a generic lower dimensional manifold remains mostly unexplored, with [13] considering the case of a set of straight lines as constraint.

In this paper, we introduce a coverage control algorithm that addresses the problem of how to cover a closed and convex planar domain, $\mathcal{D} \subset \mathbb{R}^2$, with a team of N agents constrained to move on a smooth curve defined in $\mathcal{D}$. We consider the coverage objective of the team with respect to the density function in the two-dimensional environment as in [14], but reformulate the locational cost to include the constraint that confines the robots to move on a one-dimensional manifold. A convex relaxation is then introduced such that the problem can be solved efficiently in a decentralized fashion.

The remainder of the paper is organized as follows: In Section II, we briefly introduce the coverage control problem from [14] and highlight the problems that arise when dealing with motion constraints. In Section III we present the formulation of the coverage problem for planar robots that are constrained to move on curves. A decentralized algorithm to minimize a locational cost is proposed, which leverages a convex relaxation of the cost. Section IV illustrates the application of the derived algorithm on a team of ground mobile robots tasked with environment surveilling.

II. COVERAGE CONTROL

A. Unconstrained Locational Optimization

The coverage problem for mobile sensors networks can be formulated as the deployment of a set of N mobile planar sensors in a closed and convex domain $\mathcal{D} \subset \mathbb{R}^2$ according to an information density function $\phi : \mathcal{D} \to [0, \infty)$ [14].

When monitoring the domain $\mathcal{D}$, a common strategy is to assign to each agent the surveillance of the points in the domain that are closest to it. If we let the position of Robot i in the team be denoted by $p_i \in \mathcal{D}$, $i \in \{1, \dots, N\}$, then Robot i has the responsibility of covering the set $\mathcal{V}_i(P) = \{q \in \mathcal{D} \mid \|q - p_i\| \leq \|q - p_j\|, j \in \mathcal{N}\}$, with

$P = \{p_1, \ldots, p_N\}$ being the positions of the agents and $\mathcal{N} = \{1, \ldots, N\}$ the index set. The partition $\{\mathcal{V}_1(P), \ldots, \mathcal{V}_N(P)\}$ constitutes a Voronoi partition of the domain $\mathcal{D}$ under the Euclidean metric [21].

Having established the partition of the space, each agent can evaluate the coverage performance over its region of dominance with respect to the density function, ϕ, which may represent the information density spread over the domain. Considering an isotropic sensing performance that degrades with distance, a natural choice is to penalize the coverage of a point based on the distance to the robot in charge of covering it, that is, the further the point from the robot in charge, the worse the coverage of that point [14]. Thus, the robots' performance can be encoded by the cost

$$\mathcal{H}(P) = \sum_{i\in\mathcal{N}} \int_{\mathcal{V}_i(P)} \|p_i - q\|^2 \phi(q)\,\mathrm{d}q, \tag{1}$$

with a lower value of the cost corresponding to a better coverage of the domain.

Having defined a performance metric, the team can evolve towards a critical point of the cost in (1) by making each robot move in the direction of the negative gradient of the cost. Given the gradient of (1) as in [14],

$$\frac{\partial\mathcal{H}}{\partial p_i}(P) = 2\int_{\mathcal{V}_i(P)} (p_i - q)^\mathsf{T}\phi(q)\,\mathrm{d}q, \tag{2}$$

each agent can evolve according to the dynamics $\dot{p}_i = -\frac{\partial\mathcal{H}}{\partial p_i}^\mathsf{T}(P) = \kappa(\rho_i - p_i)$, with $\kappa > 0$ a proportional constant, and $m_i = \int_{\mathcal{V}_i(P)} \phi(q)\,\mathrm{d}q$ and $\rho_i = \int_{\mathcal{V}_i(P)} q\phi(q)\,\mathrm{d}q/m_i$ the mass and center of mass of the Voronoi cell $\mathcal{V}_i$, respectively.

B. Constrained Locational Optimization

As discussed in the Introduction, the employment of wire-traversing robots for environmental monitoring applications can be advantageous in terms of motion planning and control as well as energy requirements. In order to perform monitoring tasks while being constrained to move on wires, we can define a constrained locational optimization problem.

In this section, we illustrate the problems that may arise while trying to minimize the cost in (1) in the case of robots constrained to move on a curve defined in the domain $\mathcal{D}$.

To this end, let

$$c : s \in I \subset \mathbb{R} \mapsto p \in \mathcal{D} \subset \mathbb{R}^2$$

be an arc-length parameterized, simple, regular and $C^2(I^\circ)$ curve, i. e., a curve that is twice continuously differentiable in the interior of the interval I and for which $\frac{dc}{ds} \neq 0,\ \forall s \in I$. We define the set

$$\mathscr{C} = \{c(s) \mid s \in I\} \subset \mathcal{D} \subset \mathbb{R}^2$$

as the set of points belonging to the curve. We will refer to c and $\mathscr{C}$ as the *curve*, making the distinction clear when required. Thus, the constrained locational optimization problem can be expressed as:

$$\begin{aligned}&\min_P\ \mathcal{H}(P)\\ &\text{s.t. } p_i \in \mathscr{C}\quad \forall i \in \mathcal{N},\end{aligned} \qquad \text{or,} \qquad \min_S\ \mathcal{H}_c(S), \tag{3}$$

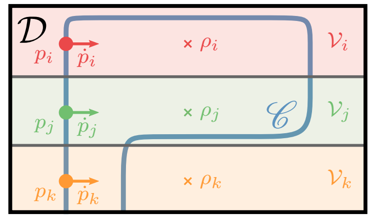


Fig. 1. Poor spatial allocation of the robots obtained by executing projected gradient descent to minimize the locational cost (1). The robots stop when their velocities $\dot{p}_i$, $\dot{p}_j$ and $\dot{p}_k$ are orthogonal to the curve $\mathscr{C}$.

where $\mathcal{H}_c(S) = (\mathcal{H}\circ c)(S) = \sum_{i\in\mathcal{N}} \int_{\mathcal{V}_i} \|c(s_i) - q\|^2\phi(q)\,\mathrm{d}q$ and $S = \{s_1, \ldots, s_N\} \subset I \times \ldots \times I$ s.t. $c(s_i) = p_i\ \forall i \in \mathcal{N}$. Assuming the robots can move according to single integrator dynamics on the curve, solving (3) using gradient descent leads to the following decentralized control law:

$$\dot{s}_i = -\frac{\partial\mathcal{H}_c}{\partial s_i}(T) = \kappa\frac{dc}{ds}^\mathsf{T}(s_i)\,(\rho_i - c(s_i)) \in \mathbb{R}, \tag{4}$$

$\kappa > 0$ a proportional constant, where chain rule has been leveraged to express $\frac{\partial\mathcal{H}_c}{\partial s_i}$ in terms of $\frac{\partial\mathcal{H}}{\partial p_i}$.

Then, the evolution in time of the cost $\mathcal{H}_c$ is given by:

$$\dot{\mathcal{H}}_c = \frac{\partial\mathcal{H}_c}{\partial s_i}\dot{s}_i = -\kappa\left|\frac{dc}{ds}^\mathsf{T}(s_i)\,(\rho_i - c(s_i))\right|^2 \leq 0. \tag{5}$$

The expression in (5) vanishes in the following cases: (i) $c(s_i) = \rho_i$, (ii) $\frac{dc}{ds}(s_i) = 0$, or (iii) $\rho_i - c(s_i) \perp \frac{dc}{ds}(s_i)$. In case (i), the position of Robot i coincides with the centroid of its Voronoi cell $\mathcal{V}_i$. Case (ii), by the regularity assumption on the curve c, by definition, can never occur. Case (iii), however, tells us that the speed of Robot i on the curve is zero when the curve is orthogonal to the line segment joining the position of Robot i and the centroid of its Voronoi cell. This condition, typical of the projected gradient descent algorithm (4), highlights the problem suffered by the constrained locational cost optimization. Figure 1 schematically depicts an example of what has been discussed in this section.

In the next section, we derive an algorithm that overcomes this problem by solving a convex approximation of the constrained optimization problem (3).

III. CONSTRAINED COVERAGE CONTROL

The discussion in Section II-B shows that, due to the non-convexity of the cost and the constraints, a gradient descent policy, albeit decentralized, can drive the robots to a stationary point corresponding to a bad spatial distribution. In this section, we show that the non-convexity of the problem is caused by the shape of the curve and that not all curves will result in non-convex problems. Indeed, under certain circumstances, (3) can actually be a convex problem. This gives insight on how to formulate the locational optimization problem for robots constrained to move on generic curves, as will be shown in the next section.

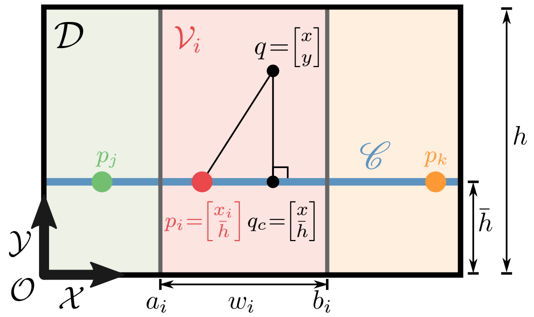


Fig. 2. Example of coverage on a straight line. The depicted quantities are used in (6) to derive the one-dimensional locational cost equivalent to (1).

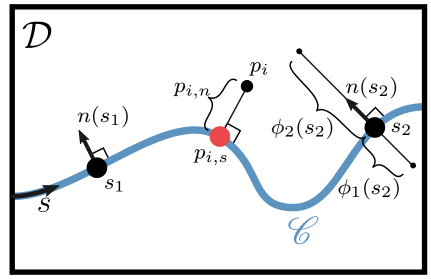


Fig. 3. Reference system $\{s, n(s)\}$ and other quantities used in the derivation of the constrained locational cost $\mathcal{H}_c(P)$.

A. Motivation

Let us consider the scenario depicted in Fig. 2. The domain $\mathcal{D}$ is a rectangle and the curve $\mathscr{C}$ is a straight line parallel to one of its sides. Assuming a constant density function $\phi(q) = 1 \; \forall q \in \mathcal{D}$, we can rewrite the cost (1) as follows:

$$\begin{aligned}\mathcal{H}(P) &= \sum_{i\in\mathcal{N}} \int_{\mathcal{V}_i(P)} \|p_i - q\|^2 \,\mathrm{d}q \\ &= \underbrace{\sum_{i\in\mathcal{N}} \int_{\mathcal{V}_i(P)} \|p_i - q_c\|^2 \,\mathrm{d}q}_{①} + \underbrace{\sum_{i\in\mathcal{N}} \int_{\mathcal{V}_i(P)} \|q - q_c\|^2 \,\mathrm{d}q}_{②},\end{aligned}$$

where $p_i = [x_i, \bar{h}]^\mathsf{T}$, $q = [x, y]^\mathsf{T}$ and $q_c = [x, \bar{h}]^\mathsf{T}$ is the projection of q onto the curve $\mathscr{C}$. Solving the integrals ① and ②, yields:

$$\begin{aligned}① &= \sum_{i\in\mathcal{N}} \int_0^h \int_{a_i}^{b_i} |x_i - x|^2 \,\mathrm{d}x\,\mathrm{d}y = h \sum_{i\in\mathcal{N}} \int_{a_i}^{b_i} |x_i - x|^2 \,\mathrm{d}x \\ &= h\mathcal{H}_c(X), \\ ② &= \sum_{i\in\mathcal{N}} \int_0^h \int_{a_i}^{b_i} |y - \bar{h}|^2 \,\mathrm{d}x\,\mathrm{d}y = \sum_{i\in\mathcal{N}} w_i \int_0^h |y - \bar{h}|^2 \,\mathrm{d}y \\ &= \sum_{i\in\mathcal{N}} w_i \left(\frac{\left(h - \bar{h}\right)^3}{3} + \frac{\bar{h}^3}{3} \right) = C, \end{aligned} \tag{6}$$

where a_i and b_i are the x coordinates of the vertical segments of the boundaries of the Voronoi cell $\mathcal{V}_i$, $w_i = |b_i - a_i|$, $\bar{h}$ is the y position of the curve $\mathscr{C}$ (see Fig. 2), $X = \{x_1, \dots, x_N\}$, and C is a constant. Thus, $\mathcal{H}(P) = h\mathcal{H}_c(X) + C$, and, therefore, $\min_P \; \mathcal{H}(P) \sim \min_X \; \mathcal{H}_c(X)$, i. e. the two minimization problems are equivalent in the following sense:

$$\left\{ \begin{bmatrix} x_1^* \\ \bar{h} \end{bmatrix}, \dots, \begin{bmatrix} x_N^* \\ \bar{h} \end{bmatrix} \right\} = \arg\min_P \; \mathcal{H}(P)$$

$$\Leftrightarrow$$

$$\{x_1^*, \dots, x_N^*\} = \arg\min_X \; \mathcal{H}_c(X).$$

In [22], it is shown that in case the domain $\mathcal{D}$ is one-dimensional and the density function ϕ is log-concave, conditions satisfied by $\mathcal{H}_c(X)$ (characterized by a constant density function), the minimization of the locational cost (1) is a convex problem. Therefore, gradient descent methods can be used to synthesize a decentralized control law that will drive the robots to a configuration corresponding to the global minimum of (1).

In the following section, we extend the construction introduced here to curves that are not necessarily straight lines.

B. Formulation for Generic Curves

Similarly to what has been done in Sec. III-A, in this section we introduce a system of coordinates, using which we evaluate the integrals that show up in the locational cost $\mathcal{H}(P)$. Fig. 3 depicts the domain $\mathcal{D}$ with a curve $\mathscr{C}$, parameterized using the arc length $s \in I_s \subset \mathbb{R}$. The system of coordinates $\{s, n(s)\}$, in which the points in the domain will be expressed, consists of the curvilinear abscissa, s (analogous to $\mathcal{X}$ in Fig. 2), and the normal to the curve at s, $n(s)$ analogous to $\mathcal{Y}$ in Fig. 2). Fig. 3 shows an example of such coordinates for a point p_i in the domain.

Proceeding as in Sec. III-A, using the system of coordinates just defined, we can write:

$$\begin{aligned}\mathcal{H}(P) &= \sum_{i\in\mathcal{N}} \int_{\mathcal{V}_i(P)} \|p_i - q\|^2 \,\mathrm{d}q \\ &\le \sum_{i\in\mathcal{N}} \int_{S_i} \int_{-\phi_1(s)}^{\phi_2(s)} \left(|p_{i,s} - s|^2 + |p_{i,n} - n|^2 \right) \mathrm{d}s\,\mathrm{d}n \\ &= \mathcal{H}_c(P), \end{aligned} \tag{7}$$

where $p_i = [p_{i,s}, p_{i,n}]^\mathsf{T}$, $S_i = \{s \in I_s \mid c(s) \in \mathcal{V}_i\}$ is a union of closed intervals, corresponding to the curve segments that lie in the Voronoi cell $\mathcal{V}_i$, and the functions ϕ_1 and ϕ_2, whose values at s_2 are shown in Fig. 3, will be defined later in the section. $\mathcal{H}_c(P)$ in (7) is analogous to (6) and, as a matter of fact, it is its generalization to non-straight curves.

Since the goal is that of minimizing the upper bound of $\mathcal{H}(P)$ in (7), we make the following assumption on the curve c in order to bound the same integral from below too, and have a well-defined optimization problem.

Assumption 1. *The curve c is such that:*

$$\begin{cases} k_1 = \max\limits_{u,v\in\mathscr{C}} \left(\widehat{uv}^2 - \|u - v\|^2 \right) < \infty \\ k_2 = \max\limits_{\substack{u\in\mathscr{C}\\ v\in\mathcal{D}}} \left(\|u - v\|_c^2 - \|u - v\|^2 \right) < \infty, \end{cases}$$

where $\widehat{uv}$ denotes the arc length between the two points u and v on the curve, whereas $\|u - v\|_c^2 \triangleq \widehat{uv_c}^2 + \|v - v_c\|^2$,

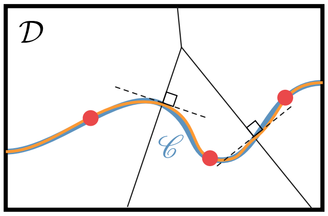


Fig. 4. An example of deformation of the curve $\mathscr{C}$ in order to fulfill Assumption 2. The actual curve is depicted in blue, while its deformation is shown in orange. The Voronoi cells generated by the robots (red dots) are shown in black.

v_c being the point on the curve closest to v characterized by the smallest curvilinear abscissa. The symbol $\|\cdot\|_c$ is an abuse of notation, since it does not define a norm (nor even a metric) as the triangle inequality does not hold. Note that $k_2 \geq k_1 \ \forall u \in \mathscr{C}, \ \forall v \in \mathcal{D}$.

Under Assumption 1, we can write:

$$-\infty < \mathcal{H}_c(P) - k_2|\mathcal{D}| \leq \mathcal{H}(P) \leq \mathcal{H}_c(P) - k_1|\mathcal{D}| < \infty,$$

where $|\mathcal{D}|$ denotes the measure of the set $\mathcal{D}$, therefore the cost $\mathcal{H}(P)$ is bounded above and below, as desired.

Now, in order to rigorously define ϕ_1 and ϕ_2, we need to introduce an additional assumption on the curve $\mathscr{C}$.

Assumption 2. *The curve $\mathscr{C}$ intersects the boundary of the domain $\mathcal{D}$ and of the Voronoi cells $\mathcal{V}_i, \ i \in \mathcal{N}$ at right angle, i. e. at the intersection points, the tangent to the curve is orthogonal to the boundary of the Voronoi cells.*

Remark 3. *Any smooth curve $c(s)$ can be continuously modified (e. g. using bump functions [23]) in an arbitrarily small neighborhood of the intersection points, in order to satisfy the condition stated in Assumption 2. Figure 4 shows an example of such a modification. Moreover, note that this construction does not need to happen on the physical curve on which the robots are constrained, since it is only required in order to calculate the control inputs to the robots.*

Under Assumption 2, the following fact holds.

Fact 4. *Let*

$$\Phi(t) = \left\{ p \in \mathcal{V}_i \mid \|p - c(t)\| \leq \|p - c(s)\| \ \forall s \in S_i \right\} \subset \mathbb{R}^2.$$

Then, under Assumption 2, $|\Phi(t)| = 0$. Moreover, $\Phi(t)$ is a segment of straight line orthogonal to the curve $\mathscr{C}$ at $c(t)$.

Fact 4 is required in order for $\phi_1(s)$, $\phi_2(s)$ and consequently the integrals in (7), to be well-defined. From Fact 4 it also follows that $\Phi(t) \subset \operatorname{span}\{n(t)\}$. Therefore, we can use the following arc-length-parameterized line segment to describe the set $\Phi(t)$:

$$\phi_t : l \in \mathbb{R} \mapsto c(t) + l \ n(t) \in \mathbb{R}^2.$$

We are now ready to define $\phi_1(s)$ and $\phi_2(s)$ as follows:

$$\phi_1 : s \in S_i \mapsto \max_l \left\{ \|\phi_s(l) - \phi_s(0)\| \mid \phi_s(l) \in \mathcal{V}_i, \ l \leq 0 \right\}$$
$$\phi_2 : s \in S_i \mapsto \max_l \left\{ \|\phi_s(l) - \phi_s(0)\| \mid \phi_s(l) \in \mathcal{V}_i, \ l \geq 0 \right\}.$$

Since $\Phi(t)$ is a line segment in $\mathbb{R}^2$, $|\Phi(t)| = 0$, and so the integrals in (7) have a geometric meaning that is entirely analogous to that of (6) in Sec. III-A.

Observing that $p_{i,n} = 0$, we can simplify (7) as follows:

$$\mathcal{H}_c(P) = \sum_{i \in \mathcal{N}} \int_{S_i} \int_{-\phi_1(s)}^{\phi_2(s)} \left(|p_{i,s} - s|^2 + |p_{i,n} - n|^2 \right) \mathrm{d}s \, \mathrm{d}n$$
$$= \sum_{i \in \mathcal{N}} \int_{S_i} \left(|p_{i,s} - s|^2 \phi(s) + \bar{\phi}(s) \right) \mathrm{d}s, \quad (8)$$

where $\phi(s) = \phi_1(s) + \phi_2(s)$ and $\bar{\phi}(s) = \frac{\phi_1(s)^3 + \phi_2(s)^3}{3}$.

As mentioned above, in case of one-dimensional domain and log-concave density function, the locational cost minimization is a convex problem [22]. The functions $\phi(s)$ and $\bar{\phi}(s)$, however, depend on the domain $\mathcal{D}$, on the position of the curve $\mathscr{C}$ in the domain and, at each time instant, on the position of the robots along the curve, through the boundaries of the Voronoi cells. Therefore, in general, $\phi(s)$ is not a log-concave function.

In the following, Proposition 7 will give the expression of a convex relaxation of the one-dimensional coverage problem, given any, not necessarily log-concave, density function. Using this, we will derive a decentralized algorithm that minimizes $\mathcal{H}_c(P)$ in (8).

C. Convex Relaxation of Constrained Coverage Control Formulation for Generic Curves

In order to formulate the convex relaxation problem, we start by observing the following.

Observation 5. *If $\mathcal{I} \subset \mathbb{R}$ is compact, $\{\mathcal{V}_1, \ldots, \mathcal{V}_N\}$ is the Voronoi partition of $\mathcal{I}$ generated by the points $P = \{p_1, \ldots, p_N\} \in \mathcal{I} \times \ldots \times \mathcal{I} = \mathcal{I}^N$, then*

$$\mathcal{H}(P) = \sum_{i \in \mathcal{N}} \int_{\mathcal{V}_i(P)} |p_i - q|^2 \phi(q) \, \mathrm{d}q \quad (9)$$

is measurable, and hence square-integrable, on $\mathcal{I}^N$.

Let us define $\mathcal{F} = \{f \mid f : \mathbb{R} \to \mathbb{R}_+\}$ as the space of functions that map real numbers to positive real numbers, and let

$$\mathscr{F} : \mathcal{F} \to L^2(\mathcal{I}^N) : \theta \mapsto \sum_{i \in \mathcal{N}} \int_{\mathcal{V}_i(P)} |p_i - q|^2 \theta(q) \, \mathrm{d}q \quad (10)$$

be the mapping that associates to each function $\theta \in \mathcal{F}$ the coverage cost (which belongs to $L^2(\mathcal{I}^N)$ as shown in Observation 5) with density function θ. In (10), $p_i, q \in \mathcal{I} \subset \mathbb{R}$, $\mathcal{I}$ compact and $\{\mathcal{V}_1, \ldots, \mathcal{V}_N\}$ is the Voronoi partition of $\mathcal{I}$ generated by $P = \{p_1, \ldots, p_N\} \in \mathcal{I}^N$. Using this notation, $\mathcal{H}(P)$ in (9) can be expressed as $\mathscr{F}(\phi)$.

Now, denoting with $\mathcal{C} = \{f \in \mathcal{F} \mid f \text{ is concave}\} \subset \mathcal{F}$ the set of concave functions that map from $\mathbb{R}$ to $\mathbb{R}_+$, we are ready to state the convex relaxation problem as follows. The convex cost, obtained by deriving a convex relaxation of (8), is given by the solution to the following program:

$$\min_{\substack{\theta \in \mathcal{C} \\ \theta > \phi}} \|\mathscr{F}(\theta) - \mathscr{F}(\phi)\|^2_{L^2(\mathcal{I}^N)}. \quad (11)$$

We insist on θ being pointwise greater than ϕ since, this way, $\mathscr{F}(\theta) \geq \mathscr{F}(\phi)\ \forall P$, and we preserve the upper bound on the two-dimensional locational cost, initially stated in (7).

The last notion we need in order to formulate the expression of a convex relaxation of the one-dimensional coverage control problem is given in the following definition.

Definition 6 (Concave envelope)**.** *Let $f : X \to \mathbb{R}$ be a real-valued function defined over the non-empty convex set $X \subset \mathbb{R}^n$. The function $g : X \to \mathbb{R}$ is the concave envelope of f over X, denoted by* $\operatorname{conc}(f)$, *if*

i) g is concave over X,
ii) $g(x) \geq f(x)\ \forall x \in X$,
iii) $g(x) \leq h(x)\ \forall x \in X$, $\forall h$ concave s.t. $h \geq f$ in X.

Proposition 7. *The following is a convex relaxation of the problem of minimizing the one-dimensional locational optimization* (9)*:*

$$\min_P\ \mathscr{F}(\operatorname{conc}(\phi)).$$

Proof. Expanding the L^2 norm in (11), one has:

$$\|\mathscr{F}(\theta) - \mathscr{F}(\phi)\|^2_{L^2(\mathcal{I}^N)} = \int_{\mathcal{I}^N} |\mathscr{F}(\theta) - \mathscr{F}(\phi)|^2$$
$$= \int_{\mathcal{I}^N} \left| \sum_{i\in\mathcal{N}} \int_{\mathcal{V}_i(P)} |p_i - q|^2 \underbrace{(\theta(q) - \phi(q))}_{\circledast} \mathrm{d}q \right|^2. \tag{12}$$

As the term $\circledast$ in (12) is positive, the minimum of $\|\mathcal{F}(\theta) - \mathcal{F}(\phi)\|^2_{L^2(\mathcal{I}^N)}$ is achieved when the difference $\theta(q) - \phi(q)$ is minimized. As θ has to be concave and $\theta > \phi$, by Definition 6, $\theta = conc(\phi)$ minimizes (12). □

With the result stated in Proposition 7, a decentralized algorithm that allows the robots to minimize the cost defined in (8) is derived in the following section.

D. A Decentralized Algorithm for Constrained Coverage Control

Starting from (8), we can write:

$$\mathcal{H}_c(P) = \sum_{i\in\mathcal{N}} \int_{S_i} |p_{i,s} - s|^2 \phi(s)\,\mathrm{d}s + \sum_{i\in\mathcal{N}} \int_{S_i} \bar{\phi}(s)\,\mathrm{d}s$$
$$= F(P, \phi) + G(P)$$

By Proposition 7, the problem of solving

$$\min_P\ \mathcal{H}_c(P) = \min_P\ \ (F(P,\phi) + G(P))$$

can be relaxed into

$$\min_P\ \ (F(P, \operatorname{conc}(\phi)) + G(P, \psi)), \tag{13}$$

where

$$F(P, \operatorname{conc}(\phi)) = \sum_{i\in\mathcal{N}} \int_{S_i} |p_{i,s} - s|^2 (\operatorname{conc}(\phi))(s)\,\mathrm{d}s \tag{14}$$

is convex, and

$$G(P, \psi) = \sum_{i\in\mathcal{N}} \int_{S_i} \left(|p_{i,s} - s|^2 \psi(s) + \bar{\phi}(s)\right)\,\mathrm{d}s, \tag{15}$$

with $\psi(s) = \phi(s) - (\operatorname{conc}(\phi))(s)$ differentiable.

The following theorem provides an algorithm for solving optimization problems where the cost function is the sum of two terms: one convex (but not necessarily differentiable) and the other differentiable (but not necessarily convex).

Theorem 8 (Theorem 3.1 in [24])**.** *Consider the minimization problem:*

$$\min_x\ h(x) = \min_x (f(x) + g(x)),$$

where $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a convex, but not necessarily differentiable, function and $g : \mathbb{R}^n \to (-\infty, +\infty]$ is a function which is continuously differentiable on an open set containing the domain of f, but g need not be convex. Then, the following algorithm is designed to generate critical points of h:

Step 1: Set $k = 0$ and initialize $x^{(k)}$ with x_0.
Step 2: Solve the convex optimization problem

$$\tilde{x}^{(k)} = \arg\min_x \left(f(x) + \left.\frac{\partial g}{\partial x}\right|_{x=x^{(k)}} x \right). \tag{16}$$

Step 3: Find

$$x^{(k+1)} = x^{(k)} + \lambda^{(k)} d^{(k)}, \tag{17}$$

with $\lambda^{(k)} > 0$, such that

$$h\left(x^{(k+1)}\right) \leq h\left(x^{(k)} + \lambda^{(k)} d^{(k)}\right), \tag{18}$$

where

$$d^{(k)} = \tilde{x}^{(k)} - x^{(k)}. \tag{19}$$

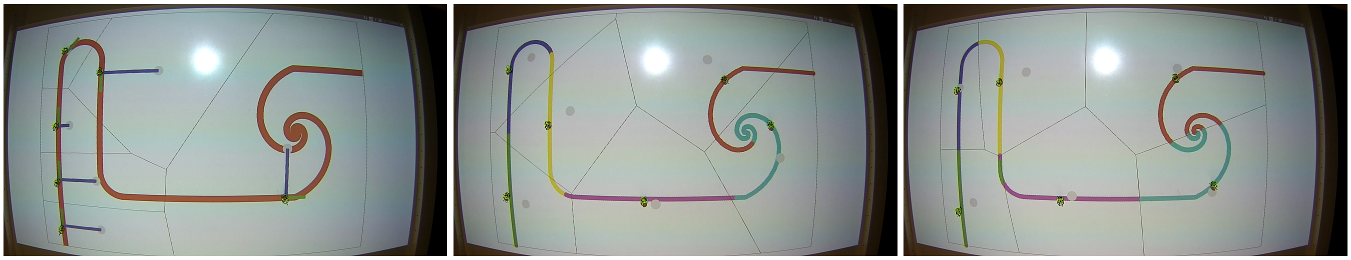
Step 4: Check for convergence: $\|d^{(k)}\| < \epsilon$, where ϵ is some prescribed positive number.
Step 5: k = k + 1

The functions $F(P, \operatorname{conc}(\phi))$ and $G(P, \psi)$ in (13), satisfy the conditions of f and g in Theorem 8. Thus, the previous algorithm can be used to find minimum points of $\mathcal{H}_c(P)$.

The calculation of $\frac{\partial G}{\partial P}$ is decentralized, hence the value of $\tilde{P}^{(k)}$ can be obtained in a decentralized way using (16). The same holds for Step 3. A more careful analysis is required to understand whether the inequality (18) can be checked in a decentralized way. Given the expressions of $F(P, \operatorname{conc}(\phi))$ and $G(P, \psi)$ in (14) and (15), we can write $\mathcal{H}_c(P) = \sum_{i\in\mathcal{N}} \mathcal{H}_{c,i}(P)$, therefore $\mathcal{H}_c(P) \geq \mathcal{H}_{c,i}(P) \geq 0,\ \forall i \in \mathcal{N}$. Thus, if each robot ensures that $\mathcal{H}_{c,i}\left(p_i^{(k+1)}\right) \leq \mathcal{H}_{c,i}\left(p_i^{(k+1)} + \lambda_i^{(k)} d_i^{(k)}\right)$, then $\mathcal{H}_c\left(P^{(k+1)}\right) \leq \mathcal{H}_c\left(P^{(k)}\right)$. This means that condition (18) can be checked in a decentralized fashion.

Assumption 9. *As is the case of the unconstrained coverage control in* (1)*, it is assumed that the robots know the density function over the domain.*

Algorithm 1 summarizes what has been derived in Sections III-C and III-D, by describing a decentralized strategy to perform constrained coverage control.



(a) Projected gradient descent: in the final configuration, the vectors pointing from the robots to the centroids of their Voronoi cell (blue) are orthogonal to the tangent to the curve (green).

(b) One-dimensional coverage: the robots consider the curve itself as the domain in the coverage problem formulation, instead of the two-dimensional environment.

(c) Algorithm 1: the spatial allocation reached by the robots corresponds to a lower locational cost, as can be seen in Fig. 6.

Fig. 5. Comparison of the final allocation of 6 small-scale differential drive robots obtained using three different coverage control algorithms on the same curve. The Voronoi cells (black thin lines), together with their centroids (gray dots), are superimposed on the three plots.

Algorithm 1 Constrained Coverage Control

Require: $\epsilon > 0$, $\gamma > 0$

initialize $k = 0$

initialize $p_i^{(k)}$ to Robot i's initial position

repeat

  measure $p_j^{(k)},\ j \in \mathcal{N}_i$

  compute Voronoi cell $\mathcal{V}_i$

  deform curve to be orthogonal to the boundary of $\mathcal{V}_i$

  calculate $p_i^{(k+1)}$ and $d^{(k)}$ ▷ (16), (17), (19)

  execute $u_i = \gamma \left(p_i^{(k+1)} - p_i^{(k)} \right)$

  $k \leftarrow k + 1$

until $\|d(k)\| < \epsilon$

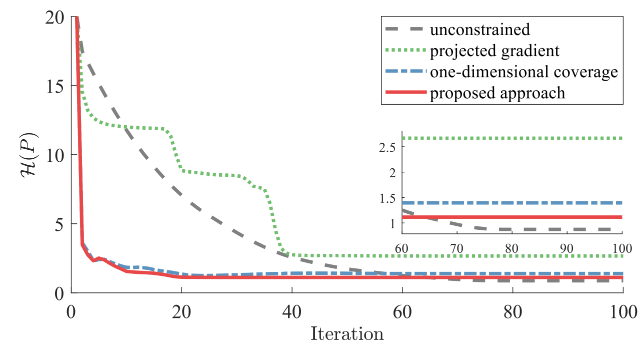


Fig. 6. Evolution of the cost in (1) when executing projected gradient descent, one-dimensional coverage and the proposed approach in Algorithm 1.

IV. EXPERIMENTS

Algorithm 1 is implemented on a team of 6 differential drive robots in the Robotarium, a remotely-accessible swarm-robotics testbed [25]. The team has to cover a 2.8m×2m rectangular area while being constrained to move on a curve, projected onto the testbed using an overhead projector (see Fig. 5). The robots act as generators for the Voronoi partition, with the black lines depicting the cell boundaries and the gray dots their centroids: the closer the robots are to their Voronoi cell centroid, the smaller the cost. In Fig. 5a the curve on which the robots move is depicted in red, whereas in Figs. 5b and 5c, it is painted using a different color for each Voronoi cell on the (one-dimensional) curve.

The proposed algorithm is compared to two other strategies for constrained coverage, i. e., projected gradient and one-dimensional coverage. Fig. 5a depicts the final allocation attained by projecting the negative gradient in (2) onto the curve. This highlights the issue brought up in Sec. II, which causes the robots to stop at an unfavorable spatial allocation given that the gradients are perpendicular to the curve. As an alternative, the curve itself is considered as the domain to be covered in Fig. 5b. Although the resulting optimization problem is convex, as discussed before, no information about the two-dimensional environment is used to optimize the locations of the robots. Thus, the performance in the two-dimensional environment is heavily influenced by the curve. Finally, Fig. 5c depicts the configuration achieved by running Algorithm 1. As can be seen, the robots end up closer to the Voronoi centroids than in the other two strategies, thus attaining a smaller value of the cost, as shown in Fig. 6. Only the unconstrained case, in which the robots are allowed fo move freely, outperforms Algorithm 1.

Although, in general, the results depend on the particular choice of the curve $\mathscr{C}$ and on its position in the domain $\mathcal{D}$, both projected gradient descent and one-dimensional coverage suffer from the aforementioned problems, which make them undesirable for real applications.

V. CONCLUSIONS

In this paper, we presented a solution to the constrained coverage control problem. A team of robots was tasked with covering a two-dimensional domain, while being constrained to move along a curve. We showed that a direct projection of the unconstrained coverage algorithm can result in poor spatial allocations of the robots. For this reason, a modification to the unconstrained locational cost was proposed to take into account the constraint introduced by the curve on which the robots move. We developed a convex relaxation to efficiently, even though approximately, solve the constrained coverage problem. The decentralized algorithm was implemented on a team of ground mobile robots, showing that the proposed approach outperforms projected gradient descent and one-dimensional coverage control.

References

[1] R. D'Andrea, "Guest editorial: A revolution in the warehouse: A retrospective on kiva systems and the grand challenges ahead," *IEEE Transactions on Automation Science and Engineering*, vol. 9, no. 4, pp. 638–639, Oct 2012.

[2] P. Weller, "Stansted airport development: a case study," *International Journal of Project Management*, vol. 6, no. 3, pp. 133 – 139, 1988.

[3] M. Ranau, *Planning Approach for Dimensioning of Automated Traffic Areas at Seaport Container Terminals*. New York, NY: Springer New York, 2011, pp. 179–193.

[4] J. Evans, B. Krishnamurthy, B. Barrows, T. Skewis, and V. Lumelsky, "Handling real-world motion planning: a hospital transport robot," *IEEE Control Systems Magazine*, vol. 12, no. 1, pp. 15–19, Feb 1992.

[5] K. Toussaint, N. Pouliot, and S. Montambault, "Transmission line maintenance robots capable of crossing obstacles: State-of-the-art review and challenges ahead," *Journal of Field Robotics*, vol. 26, no. 5, pp. 477–499, Mar 2009.

[6] M. Rahimi, R. Pon, W. J. Kaiser, G. S. Sukhatme, D. Estrin, and M. Srivastava, "Adaptive sampling for environmental robotics," in *IEEE International Conference on Robotics and Automation, 2004. Proceedings. ICRA '04. 2004*, vol. 4, April 2004, pp. 3537–3544 Vol.4.

[7] J. Billingsley, A. Visala, and M. Dunn, *Robotics in Agriculture and Forestry*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008, pp. 1065–1077.

[8] M. Dunbabin and L. Marques, "Robots for environmental monitoring: Significant advancements and applications," *IEEE Robotics & Automation Magazine*, vol. 19, no. 1, pp. 24–39, 2012.

[9] M. Nayyerloo, X. Chen, W. Wang, and J. G. Chase, "Cable-climbing robots for power transmission lines inspection," in *Mobile Robots*, X. Chen, Y. Chen, and J. Chase, Eds. Rijeka: IntechOpen, 2009, ch. 4.

[10] J. Mampel, K. Gerlach, C. Schilling, and H. Witte, "A modular robot climbing on pipe-like structures," in *2009 4th International Conference on Autonomous Robots and Agents*, Feb 2000, pp. 87–91.

[11] P. Debenest, M. Guarnieri, K. Takita, E. F. Fukushima, S. Hirose, K. Tamura, A. Kimura, H. Kubokawa, N. Iwama, and F. Shiga, "Expliner - robot for inspection of transmission lines," in *2008 IEEE International Conference on Robotics and Automation*, May 2008, pp. 3978–3984.

[12] S. Aoshima, T. Tsujimura, and T. Yabuta, "A wire mobile robot with multi-unit structure," in *Proceedings. IEEE/RSJ International Workshop on Intelligent Robots and Systems '. (IROS '89) 'The Autonomous Mobile Robots and Its Applications*, Sept 1989, pp. 414–421.

[13] G. Notomista and M. Egerstedt, "Coverage control for wire-traversing robots," in *2018 IEEE International Conference on Robotics and Automation (ICRA)*, May 2018, pp. 1–6.

[14] J. Cortes, S. Martinez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Transactions on Robotics and Automation*, vol. 20, no. 2, pp. 243–255, 2004.

[15] A. Breitenmoser, M. Schwager, J. Metzger, R. Siegwart, and D. Rus, "Voronoi coverage of non-convex environments with a group of networked robots," in *2010 IEEE International Conference on Robotics and Automation*, May 2010, pp. 4982–4989.

[16] Y. Stergiopoulos, M. Thanou, and A. Tzes, "Distributed collaborative coverage-control schemes for non-convex domains," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2422–2427, Sept 2015.

[17] S. Bhattacharya, R. Ghrist, and V. Kumar, "Multi-robot coverage and exploration on riemannian manifolds with boundaries," *The International Journal of Robotics Research*, vol. 33, no. 1, pp. 113–137, 2014.

[18] A. Pierson, L. C. Figueiredo, L. C. Pimenta, and M. Schwager, "Adapting to sensing and actuation variations in multi-robot coverage," *The International Journal of Robotics Research*, vol. 36, no. 3, pp. 337–354, 2017.

[19] O. Arslan and D. E. Koditschek, "Voronoi-based coverage control of heterogeneous disk-shaped robots," in *2016 IEEE International Conference on Robotics and Automation (ICRA)*, May 2016, pp. 4259–4266.

[20] M. Santos, Y. Diaz-Mercado, and M. Egerstedt, "Coverage control for multirobot teams with heterogeneous sensing capabilities," *IEEE Robotics and Automation Letters*, vol. 3, no. 2, pp. 919–925, April 2018.

[21] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu, *Spatial tessellations: concepts and applications of Voronoi diagrams*. John Wiley & Sons, 2009, vol. 501.

[22] J. Kieffer, "Uniqueness of locally optimal quantizer for log-concave density and convex error weighting function," *IEEE Transactions on Information Theory*, vol. 29, no. 1, pp. 42–47, 1983.

[23] L. W. Tu, "Bump functions and partitions of unity," *An Introduction to Manifolds*, pp. 127–134, 2008.

[24] H. Mine and M. Fukushima, "A minimization method for the sum of a convex function and a continuously differentiable function," *Journal of Optimization Theory and Applications*, vol. 33, no. 1, pp. 9–23, 1981.

[25] D. Pickem, P. Glotfelter, L. Wang, M. Mote, A. Ames, E. Feron, and M. Egerstedt, "The robotarium: A remotely accessible swarm robotics research testbed," in *Robotics and Automation (ICRA), 2017 IEEE International Conference on*. IEEE, 2017, pp. 1699–1706.